Tensor and spin representations of SO(4) and discrete quantum gravity

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Abstract. Starting from the defining transformations of complex matrices for the SO(4) group, we construct the fundamental representation and the tensor and spinor representations of the group SO(4). Given the commutation relations for the corresponding algebra, the unitary representations of the group in terms of the generalized Euler angles are constructed. These mathematical results help us to a more complete description of the Barrett-Crane model in Quantum Gravity. In particular a complete realization of the weight function for the partition function is given and a new geometrical interpretation of the asymptotic limit for the Regge action is presented.

Keywords: SO(4) group, tensor representation, spin representation, quantum gravity, spin networks.

1. Discrete models in quantum gravity

The use of discrete models in Physics has become very popular, mainly for two reasons. It helps to find the solutions of some differential equations by numerical methods, which would not be possible to solve by analytic methods. Besides that, the introduction of a lattice is equivalent to the introduction of a cut-off in the momentum variable for the field in order to achieve the finite limit of the solution. In the case of relativistic field equations -like the Dirac, Klein-Gordon, and the electromagnetic interactions- we have worked out some particular cases [1].

There is an other motivation for the discrete models and it is based in some philosophical presuppositions that the space-time structure is discrete. This is more attractive in the case of general relativity and quantum gravity because it makes more transparent the connection between the discrete properties of the intrinsic curvature and the background independent gravitational field.

This last approach was started rigorously by Regge in the early sixties [2]. He introduces some triangulation in a Riemannian manifold, out of which he constructs local curvature, coordinate independent, on



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the polyhedra. With the help of the total curvature on the vertices of the discrete manifold he constructs a finite action which, in the continuous limit, becomes the standard Hilbert-Einstein action of general relativity.

Regge himself applied his method ("Regge calculus") to quantum gravity in three dimensions [3]. In this work he assigns some representation of the SU(2) group to the edges of the triangles. To be more precise, to every tetrahedron appearing in the discrete triangulation of the manifold he associates a 6j-symbol in such a way that the spin eigenvalues of the corresponding representation satisfy sum rules described by the edges and vertices of the tetrahedra. Since the value of the 6j-symbol has a continuous limit when some edges of the tetrahedra become very large, he could calculate the sum of this limit for all the 6j-symbols attached to the tetrahedra, and in this way he could compare it with the continuous Hilbert-Einstein action corresponding to an Euclidean non planar manifold.

A different approach to the discretization of space and time was taken by Penrose [4]. Given some graph representing the interaction of elementary units satisfying the rules of angular momentum without an underlying space, he constructs out of this network ("spin network") the properties of total angular momentum as a derived concept. Later this model was applied to quantum gravity in the sense of Ponzano and Regge. In general, a spin network is a triple (γ, ρ, i) where γ is a graph with a finite set of edges e, a finite set of vertices v, ρ_e is the representation of a group G attached to an edge, and i_v is an intertwiner attached to each vertex. If we take the product of the amplitudes corresponding to all the edges and vertices (given in terms of the representations and intertwiners) we obtain the particular diagram of some quantum state.

Although the physical consequences of Penrose's ideas were soon considered to be equivalent to the Ponzano-Regge approach to quantum gravity [5], the last method was taken as guiding rule in the calculation of partition functions. We can mention a few results. Turaev and Viro [6] calculated the state sum for a 3d-triangulated manifold with tetrahedra described by 6j-symbols using the $SU(2)_q$ group. This model was enlarged to 4-dimensional triangulations and was proved by Turaev, Oguri, Crane and Yetter [7] to be independent of the triangulation (the "TOCY model").

A different approach was introduced by Boulatov [8] that led to the same partition function as the TOCY model, but with the advantage that the terms corresponding to the kinematics and the interaction could be distinguished. For this purpose he introduced some fields defined over the elements of the groups SO(3), invariant under the action of the group, and attached to the edges of the tetrahedra. The

kinematical term corresponds to the self interacting field over each edge and the interaction term corresponds to the fields defined in different edges and coupled among themselves. This method (the Boulatov matrix model) was very soon enlarged to 4-dimensional triangulations by Ooguri [9]. In both models the fields over the matrix elements of the group are expanded in terms of the representations of the group and then integrated out, with the result of a partition function extended to the amplitudes over all tetrahedra, all edges and vertices of the triangulation.

A more abstract approach was taken by Barrett and Crane, generalizing Penrose's spin networks to 4 dimensions. The novelty of this model consists in the association of representation of the SO(4) group to the faces of the tetrahedra. We will come back to this model in section 5.

Because we are interested in the physical and mathematical properties of the Barrett-Crane model, we mention briefly some recent work about this model combined with the matrix model approach of Boulatov and Ooguri [10]. In this work the 2d– quantum space-time emerges as a Feynman graph, in the manner of the 4d– matrix models. In this way a spin foam model is connected to the Feynman diagram of quantum gravity.

In this paper we have tried to implement all the mathematical consequences of Barrett-Crane model using the group theoretical properties of SO(4) applied to the 4d-triangulation of manifolds in terms of 4-simplices. It turns out that when we take into account the description of spin representations of SO(4) the weight function given by Barrett and Williams is incomplete; besides the values for the areas in the Regge action can be calculated in our paper directly from geometrical considerations.

2. The groups SO(4, R) and $SU(2) \times SU(2)$

The rotation group in 4 dimensions is the group of linear transformations that leaves the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ invariant. The well known fact that this group is locally isomorphic to $SU(2) \times SU(2)$ enables one to decompose the group action in the following way:

Take a complex matrix (not necessarily unimodular)

$$w = \begin{pmatrix} y & z \\ -\bar{z} & \bar{y} \end{pmatrix} \quad , \quad y = x_1 + ix_2, -\bar{z} = x_3 + ix_4, \tag{1}$$

where w satisfies $w w^+ = \det(w)$.

We define the *complete* group action

$$w \to w' = u_1 w u_2, \tag{2}$$

where $u_1, u_2 \in SU(2)$ correspond to the left, right action, respectively,

$$u_1 = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)^L , \quad \alpha\bar{\alpha} + \beta\bar{\beta} = 1,$$

$$u_2 = \begin{pmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{pmatrix} \in SU(2)^R , \quad \gamma\bar{\gamma} + \delta\bar{\delta} = 1.$$

The complete group action satisfies:

$$w'w'^{+} = \det(w') = ww^{+} = \det(w),$$
 (3)

or ${x_1'}^2 + {x_2'}^2 + {x_3'}^2 + {x_4'}^2 = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$, which corresponds to the defining relation for SO(4,R).

In order to make connection with R^4 , we take only the *left* action $w' = u_1 w$ and express the matrix elements of w as a 4-vector

$$\begin{pmatrix} y' \\ -\bar{z}' \\ z' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\bar{\beta} & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} y \\ -\bar{z} \\ z \\ \bar{y} \end{pmatrix}. \tag{4}$$

Substituting $y=x_1+ix_2$, $-\bar{z}=x_3+ix_4$, and $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$, we get

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \alpha_1 & -\alpha_2 & \beta_1 & -\beta_2 \\ \alpha_2 & \alpha_1 & \beta_2 & \beta_1 \\ -\beta_1 & -\beta_2 & \alpha_1 & \alpha_2 \\ \beta_2 & -\beta_1 & -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{5}$$

Obviously, the transformation matrix is orthogonal. Similarly for the right action $w' = wu_2^+$ we get

$$\begin{pmatrix} y' \\ -\bar{z}' \\ z' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} \bar{\gamma} & 0 & \bar{\delta} & 0 \\ 0 & \bar{\gamma} & 0 & \bar{\delta} \\ -\delta & 0 & \gamma & 0 \\ 0 & -\delta & 0 & \gamma \end{pmatrix} \begin{pmatrix} y \\ -\bar{z} \\ z \\ \bar{y} \end{pmatrix}, \tag{6}$$

and after substituting $\gamma = \gamma_1 + i\gamma_2$, $\delta = \delta_1 + i\delta_2$, we get

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 & -\delta_1 & \delta_2 \\ -\gamma_2 & \gamma_1 & \delta_2 & \delta_1 \\ \delta_1 & -\delta_2 & \gamma_1 & \gamma_2 \\ -\delta_2 & -\delta_1 & -\gamma_2 & \gamma_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{7}$$

where the transformation matrix is orthogonal.

If we take the complete action

$$\begin{pmatrix} y' & z' \\ -\bar{z}' & \bar{y}' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} y & z \\ -\bar{z} & \bar{y} \end{pmatrix} \begin{pmatrix} \bar{\gamma} & -\delta \\ \bar{\delta} & \gamma \end{pmatrix},$$

we get

$$\begin{pmatrix} y' \\ -\bar{z}' \\ z' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} \alpha\bar{\gamma} & \beta\bar{\gamma} & \alpha\bar{\delta} & \beta\bar{\delta} \\ -\beta\bar{\gamma} & \bar{\alpha}\bar{\gamma} & -\bar{\beta}\bar{\delta} & \bar{\alpha}\bar{\delta} \\ -\alpha\delta & -\beta\delta & \alpha\gamma & \beta\gamma \\ \bar{\beta}\delta & -\bar{\alpha}\delta & -\bar{\beta}\gamma & \bar{\alpha}\gamma \end{pmatrix} \begin{pmatrix} y \\ -\bar{z} \\ z \\ \bar{y} \end{pmatrix} = \\
= \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\bar{\beta} & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \bar{\gamma} & 0 & \bar{\delta} & 0 \\ 0 & \bar{\gamma} & 0 & \bar{\delta} \\ -\delta & 0 & \gamma & 0 \\ 0 & -\delta & 0 & \gamma \end{pmatrix} \begin{pmatrix} y \\ -\bar{z} \\ z \\ \bar{y} \end{pmatrix}, (8)$$

and taking $y = x_1 + ix_2$, $-\bar{z} = x_3 + ix_4$ we get the general transformation matrix for the 4-dimensional vector in \mathbb{R}^4 under the group $SO(4, \mathbb{R})$ as

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \alpha_1 & -\alpha_2 & \beta_1 & -\beta_2 \\ \alpha_2 & \alpha_1 & \beta_2 & \beta_1 \\ -\beta_1 & -\beta_2 & \alpha_1 & \alpha_2 \\ \beta_2 & -\beta_1 & -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 & -\delta_1 & \delta_2 \\ -\gamma_2 & \gamma_1 & \delta_2 & \delta_1 \\ \delta_1 & -\delta_2 & \gamma_1 & \gamma_2 \\ -\delta_2 & -\delta_1 & -\gamma_2 & \gamma_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$
(9)

Notice that the eight parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ with the constraints $\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$, $\gamma_1^2 + \gamma_2^2 + \delta_1^2 + \delta_2^2 = 1$, can be considered the Cayley parameters for the SO(4) group [11].

3. Tensor and spinor representations of SO(4,R)

Given the fundamental 4-dimensional representation of SO(4,R) in terms of the parameters $\alpha, \beta, \gamma, \delta$, as given in (9),

$$x_{\mu}' = g_{\mu\nu} x_{\nu},\tag{10}$$

the tensor representations are defined in the usual way

$$T_{k'_1 k'_2 \dots k'_n} = g_{k'_1 k_1} \dots g_{k'_n k_n} T_{k_1 k_2 \dots k_n},$$
 (11)
 $(k'_i, k_i = 1, 2, 3, 4).$

For the sake of simplicity we take the second rank tensors. We can decompose them into totally symmetric and antisymmetric tensors, namely,

$$S_{ij} \equiv x_i y_j + x_j y_i$$
 (totally symmetric),
 $A_{ij} \equiv x_i y_j - x_j y_i$ (antisymmetric).

If we substract the trace from S_{ij} we get a tensor that transforms under an irreducible representation. For the antisymmetrie tensor the situation is more delicate. In general we have

$$A'_{ij} \equiv x'_i y'_j - x'_j y'_i = (g_{i\ell} g_{jm} - g_{j\ell} g_{im}) A_{\ell m}.$$
 (12)

This representation of dimension 6 is still reducible. For simplicity take the left action of the group given in (5). The linear combination of the antisymmetric tensor components are transformed among themselves in the following way:

$$\begin{pmatrix} A'_{12} + A'_{34} \\ A'_{31} + A'_{24} \\ A'_{23} + A'_{14} \end{pmatrix} = \begin{pmatrix} A_{12} + A_{34} \\ A_{31} + A_{24} \\ A_{23} + A_{14} \end{pmatrix},$$
(13)
$$\begin{pmatrix} A'_{12} - A'_{34} \\ A'_{31} - A'_{24} \\ A'_{23} - A'_{14} \end{pmatrix} = \begin{pmatrix} \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 & -2(\alpha_1\beta_2 - \alpha_2\beta_1) & -2(\alpha_1\beta_1 + \alpha_2\beta_2) \\ 2(\alpha_1\beta_2 + \alpha_2\beta_1) & \alpha_1^2 - \alpha_2^2 + \beta_1^2 - \beta_2^2 & 2(\alpha_1\alpha_2 - \beta_1\beta_2) \\ 2(\alpha_1\beta_1 - \alpha_2\beta_2) & -2(\alpha_1\alpha_2 + \beta_1\beta_2) & \alpha_1^2 - \alpha_2^2 - \beta_1^2 + \beta_2^2 \end{pmatrix} \times \begin{pmatrix} A_{12} - A_{34} \\ A_{31} - A_{24} \end{pmatrix}.$$
(14)

In the case of the right action given by (7) the 6-dimensional representation for the antisymmetrie second rank tensor decomposes into two irreducible 3-dimensional representation of SO(4, R). For this purpose one takes the linear combination of the components of the antisymmetric tensor as before:

$$\begin{pmatrix} A'_{23} - A'_{14} \\ A'_{31} - A'_{24} \\ A'_{12} - A'_{34} \end{pmatrix} = \begin{pmatrix} A_{23} - A_{14} \\ A_{31} - A_{24} \\ A_{12} - A_{34} \end{pmatrix},$$

$$\begin{pmatrix} A'_{23} + A'_{14} \\ A'_{31} + A'_{24} \\ A'_{12} + A'_{34} \end{pmatrix} = \begin{pmatrix} \gamma_1^2 - \gamma_2^2 - \delta_1^2 + \delta_2^2 & 2(\gamma_1 \gamma_2 + \delta_1 \delta_2) & -2(\gamma_1 \delta_1 - \gamma_2 \delta_2) \\ -2(\gamma_1 \gamma_2 - \delta_1 \delta_2) & \gamma_1^2 - \gamma_2^2 + \delta_1^2 - \delta_2^2 & 2(\gamma_1 \delta_2 + \gamma_2 \delta_1) \\ 2(\gamma_1 \delta_1 + \gamma_2 \delta_2) & -2(\gamma_1 \delta_2 - \gamma_2 \delta_1) & \gamma_1^2 + \gamma_2^2 - \delta_1^2 - \delta_2^2 \end{pmatrix} \times \begin{pmatrix} A_{23} + A_{14} \\ A_{31} + A_{24} \\ A_{12} + A_{34} \end{pmatrix}.$$

$$(15)$$

Therefore the 6-dimensional representation for the antisymmetric tensor decomposes into two irreducible 3-dimensional irreducible representation of the SO(4, R) group.

For the spinor representation of $SU(2)^L$ we take

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} , \quad a_1, a_2 \in \not\subset$$
 (17)

Let $a^{i_1 i_2 \dots i_k}$, $(i_1, i_2, \dots i_k = 1, 2)$ be a set of complex numbers of dimension 2^k which transform under the $SU(2)^L$ group as follows:

$$a^{i'_1 \dots i'_k} = u_{i'_1 i_1} \dots u_{i'_k i_k} a^{i_1 \dots i_k}, \tag{18}$$

where $u_{i'_1i_1}, u_{i'_2i_2}...$ are the components of $u \in SO(2)^L$. If $a^{i_1...i_k}$ is totally symmetric in the indices $i_1...i_k$ the representation of dimension (k+1) is irreducible. In an analogous way we can define an irreducible representation of $SU(2)^R$ with respect to the totally symmetric multispinor of dimension $(\ell+1)$.

For the general group $SU(4,R) \sim SU(2)^L \otimes SU(2)^R$ we can take a set of totaly symmetric multispinors that transform under the SO(4) group as

$$a^{i'_1 \dots i'_k j'_1 \dots j'_\ell} = u_{i'_1 i_1} \dots u_{i'_k i_k} \bar{v}_{j'_1 j_1} \dots \bar{v}_{j'_\ell i_\ell} a^{i_1 \dots i_k j_1 \dots j_\ell}$$
(19)

where $u_{i'_1i_1}$... are the components of a general element of $SU(2)^L$ and $\bar{v}_{j'_\ell i_\ell}$ are the components of a general element of $SU(2)^R$. They define an irreducible representation of SO(4,R) of dimension $(k+1)(\ell+1)$ and with labels (see next section)

$$\ell_0 = \frac{k - \ell}{2} , \quad \ell_1 = \frac{k + \ell}{2} + 1.$$
 (20)

4. Representations of the algebra so(4, R)

Let J_1, J_2, J_3 be the generators corresponding to the rotations in the planes $(x_2, x_3), (x_3, x_1)$, and (x_1, x_2) respectively, and K_1, K_2, K_3 the generators corresponding to the rotations (boost) in the planes $(x_1, x_4), (x_2, x_4)$ and (x_3, x_4) respectively. They satisfy the following commutation relations:

$$\begin{split} [J_p,J_q] &= i\varepsilon_{pqr}J_r \quad , \quad p,q,r=1,2,3, \\ [J_p,K_q] &= i\varepsilon_{pqr}K_r, \\ [K_p,K_q] &= i\varepsilon_{pqr}J_r. \end{split} \tag{21}$$

If one defines
$$\bar{A} = \frac{1}{2} (\bar{J} + \bar{K})$$
 , $\bar{B} = \frac{1}{2} (\bar{J} - \bar{K})$, with $\bar{J} = (J_1, J_2, J_3)$, $\bar{K} = (K_1, K_2, K_3)$, then
$$[A_p, A_q] = i\varepsilon_{pqr}A_r \quad , \quad p, q, r = 1, 2, 3,$$

$$[B_p, B_q] = i\varepsilon_{pqr}B_r,$$

$$[A_p, B_q] = 0,$$
 (22)

that is to say, the algebra so(4) decomposes into two simple algebras $su(2) \times su(2)$

Let $\phi_{m_1m_2}$ be a basis where \bar{A}^2, A_3 and \bar{B}^2, B_3 are diagonal. Then a unitary irreducible representation for the sets $\{A_{\pm} \equiv A_1 \pm iA_2, A_3\}$ and $\{B_{\pm} \equiv B_1 \pm iB_2, B_3\}$ is given by

$$A_{\pm}\phi_{m_1m_2} = \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\phi_{m_1\pm 1, m_2},$$

$$A_3\phi_{m_1m_2} = m_1\phi_{m_1m_2} , \quad -j_1 \le m_1 \le j_1,$$
(23)

$$B_{\pm}\phi_{m_1m_2} = \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\phi_{m_1m_2\pm 1},$$

$$B_{3}\phi_{m_1m_2} = m_2\phi_{m_1m_2} , -j_2 \le m_2 \le j_2.$$

We change now to a new basis

$$\psi_{jm} = \sum_{m_1 + m_2 = m} \langle j_1 m_1 j_2 m_2 \mid jm \rangle \,\phi_{m_1 m_2} \tag{24}$$

that corresponds to the Gelfand-Zetlin basis for so(4),

$$\psi_{jm} = \begin{vmatrix} j_1 + j_2 & , & j_1 - j_2 \\ & j & \\ & m & \end{vmatrix}.$$

In this basis the representation for the generators \bar{J}, \bar{K} of so(4) are given by [12]

$$J_{\pm}\psi_{jm} = \sqrt{(j \mp m)(j \pm m + 1)}\psi_{jm\pm 1},$$

$$J_{3}\psi_{jm} = m\psi_{jm},$$

$$K_{3}\psi_{jm} = a_{jm}\psi_{j-1,m} + b_{jm}\psi_{jm} + a_{j+1,m}\psi_{j+1,m},$$
(25)

where

$$a_{jm} \equiv \left(\frac{(j^2 - m^2)(j^2 - \ell_0^2)(\ell_1^2 - j^2)}{(2j - 1)j^2(2j + 1)}\right)^{1/2}, \ b_{jm} = \frac{m\ell_0\ell_1}{j(j + 1)},$$

with $\ell_0 = j_1 - j_2$, $\ell_1 = j_1 + j_2 + 1$ the labels of the representations.

The representation for K_1, K_2 are obtained with the help of the commutation relations.

The Casimir operators are

$$(\bar{J}^2 + \bar{K}^2) \psi_{jm} = (\ell_0^2 + \ell_1^2 - 1) \psi_{jm}, \tag{26}$$

$$\bar{J} \cdot \bar{K} \psi_{jm} = \ell_0 \ell_1 \psi_{jm}. \tag{27}$$

The representations in the bases ψ_{jm} are irreducible in the following cases

$$\ell_0 = j_1 - j_2 = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \dots,$$

$$\ell_1 = j_1 + j_2 - 1 = |\ell_0| + 1, |\ell_0| + 2, \dots,$$

$$j = |j_1 - j_2|, \dots, j_1 + j_2.$$

If we exponentiate the infinitesimal generators we obtain the finite representations of SO(4,R) given in terms of the rotation angles. An element U of SO(4,R) is given as [13]

$$U(\varphi, \theta, \tau, \alpha, \beta, \gamma) = R_3(\varphi) R_2(\theta) S_3(\tau) R_3(\alpha) R_2(\beta) R_3(\gamma), \quad (28)$$

where R_2 is the rotation matrix in the (x_1x_3) plane, R_3 the rotation matrix in the (x_1x_2) plane and S_3 the rotation ("boost") in the (x_3x_4) plane, and

$$0 \leq \beta, \psi, \theta \leq \pi \ , \ 0 \leq \alpha, \varphi, \gamma \leq 2\pi.$$

In the basis ψ_{jm} the action of S_3 is as follows:

$$S_3(\tau) \psi_{jm} = \sum_{j'} d_{j'jm}^{j_1 j_2}(\tau) \psi_{j'm},$$
 (29)

where

$$d_{j'jm}^{j_1j_2}(\tau) = \sum_{m_1m_2} \langle j_1 j_2 m_1 m_2 \mid jm \rangle e^{-i(m_1 - m_2)\tau} \langle j_1 j_2 m_1 m_2 \mid j'm \rangle \quad (30)$$

is the Biedenharn-Dolginov function [14].

From this function the general irreducible representations of the operator U in terms of rotation angles is [13]:

$$U(\varphi, \theta, \tau, \alpha, \beta, \gamma) \psi_{jm} = \sum_{j'm'} D_{j'm'jm}^{j_1j_2} (\varphi, \theta, \tau, \alpha, \beta, \gamma) \psi_{j'm'}, \qquad (31)$$

where

$$D_{j'm'jm}^{j_{1}j_{2}}(\varphi,\theta,\tau,\alpha,\beta,\gamma) = \sum_{m''} D_{m'm''}^{j'}(\varphi,\theta,0) d_{j'jm''}^{j_{1}j_{2}}(\tau) D_{m''m}^{j}(\alpha,\beta,\gamma).$$
(32)

We now give some particular values of these representations. In the case of spin j=1/2 we know

$$R_3(\alpha) R_2(\beta) R_3(\gamma) = \begin{pmatrix} \cos \frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}} & i \sin \frac{\beta}{2} e^{-i\left(\frac{\gamma-\alpha}{2}\right)} \\ i \sin \frac{\beta}{2} e^{i\frac{\gamma-\alpha}{2}} & \cos \frac{\beta}{2} e^{-i\left(\frac{\alpha+\gamma}{2}\right)} \end{pmatrix}$$

Introducing the new parameters $\alpha + \gamma = \delta$, $\gamma - \alpha = \eta$ and the variables

$$x_1 = \cos\frac{\beta}{2}\cos\frac{\delta}{2} , \quad x_2 = \cos\frac{\beta}{2}\sin\frac{\delta}{2},$$
$$x_3 = \sin\frac{\beta}{2}\sin\frac{\eta}{2} , \quad x_4 = \sin\frac{\beta}{2}\cos\frac{\eta}{2},$$

we have

$$R_3(\alpha) R_2(\beta) R_3(\gamma) = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$
 (33)

Similarly we have

$$R_3(\varphi) R_2(\theta) S_3(\tau) = \begin{pmatrix} y_1 + iy_2 & y_3 + iy_4 \\ -y_3 + iy_4 & y_1 - iy_2 \end{pmatrix},$$
 (34)

with

$$y_1 = \cos \frac{\theta}{2} \cos \frac{\varphi + \tau}{2}$$
, $y_2 = \cos \frac{\theta}{2} \sin \frac{\varphi + \tau}{2}$, $y_3 = \sin \frac{\theta}{2} \sin \frac{\tau - \varphi}{2}$, $y_4 = \sin \frac{\theta}{2} \cos \frac{\tau - \varphi}{2}$.

For the Biedenharn-Dolginov function we have some particular values [15]

$$d_{jmm}^{[j_{+},0]}(\tau) = i^{j-m} 2^{j} \sqrt{2j+1} \Gamma(j+1) \times \left(\frac{\Gamma(m+\frac{3}{2}) \Gamma(j_{+}-m+1) \Gamma(j_{+}-j+1) \Gamma(j+m+1)}{\Gamma(\frac{3}{2}) \Gamma(j_{+}+m+2) \Gamma(j_{+}+j+2) \Gamma(j-m+1) \Gamma(m+1)} \right)^{\frac{1}{2}} \times (\sin \tau)^{j-m} C_{j_{+}-j}^{j+1}(\cos \tau),$$
(35)

where $j_{+} \equiv j_{1} + j_{2}$, $j_{-} = j_{1} - j_{2} = 0$, and $C_{n}^{\nu}(\cos \tau)$ are the Gegenbauer (ultraspherical) polynomials which are related to the Jacobi polynomials by

$$C_n^{\nu}\left(\cos\tau\right) = \frac{\Gamma\left(\nu + \frac{3}{2}\right)\Gamma\left(2\nu + n\right)}{\Gamma\left(2\nu\right)\Gamma\left(\nu + n + \frac{1}{2}\right)} P_n^{\left(\nu - \frac{1}{2}, \nu - \frac{1}{2}\right)}\left(\cos\tau\right),\,$$

from which it can be deduced that

$$d_{000}^{[j_{+},0]}(\tau) = \frac{1}{j_{+}+1} \frac{\sin((j_{+}+1)\tau)}{\sin\tau}.$$
 (36)

From the asymptotic relations of $C_n^{\nu}(\cos \tau)$ it can be proved

$$d_{jmm}^{[j_{+},0]}(\tau) \xrightarrow[j_{+} \to \infty]{} \frac{i^{j-m} \cos\left[\left(j_{+}+1\right)\tau - \frac{1}{2}\left(j+1\right)\pi\right]}{j_{+}^{m+1} \frac{\left(\sin\tau\right)^{m+1}}{\left(\sin\tau\right)^{m+1}}}.$$
 (37)

5. Relativistic spin network in 4-dimensions

We address ourselves to the Barrett-Crane model that generalized Penrose's spin networks from three dimensions to four dimensions [16]. They characterize the geometrical properties of 4-simplices, out of which the tesselation of the 4-dimensional manifold is made, and then attach to them the representations of SO(4).

A geometric 4-simplex in Euclidean space is given by the embedding of an ordered set of 5 points in $R^4(0, x, y, z, t)$ which is required to be non-degenerate (the points should not lie in any hyperplane). Each triangle in it determines a bivector constructed out of the vectors for the edges. Barrett and Crane proved that classically, a geometric 4-simplex in Euclidean space is completely characterized (up to parallel translation an inversion through the origin) by a set of 10 bivectors b_i , each corresponding to a triangle in the 4-simplex and satisfying the following properties:

- i) the bivector changes sign if the orientation of the triangle is changed;
- ii) each bivector is simple, i.e. is given by the wedge product of two vectors for the edges;
- iii) if two triangles share a common edge, the sum of the two bivector is simple;
- iv) the sum (considering orientation) of the 4 bivectors corresponding to the faces of a tetrahedron is zero;
- v) for six triangles sharing the same vertex, the six corresponding bivectors are linearly independent;
- vi) the bivectors (thought of as operators) corresponding to triangles meeting at a vertex of a tetrahedron satisfy tr $b_1[b_2, b_3] > 0$ i.e. the tetrahedron has non-zero volume.

Then Barrett and Crane define the quantum 4-simplex with the help of bivectors thought as elements of the Lie algebra SO(4), associating a representation to each triangle and a tensor to each tetrahedron. The representations chosen should satisfy the following conditions corresponding to the geometrical ones:

- i) different orientations of a triangle correspond to dual representations;
- ii) the representations of the triangles are "simple" representations of SO(4), i.e. $j_1 = j_2$;
- given two triangles, if we decompose the pair of representations into its Clebsch-Gordan series, the tensor for the tetrahedron is decomposed into summands which are non-zero only for simple representations;
- iv) the tensor for the tetrahedron is invariant under SO(4).

Now it is easy to construct an amplitude for the quantum 4-simplex. The graph for a relativistic spin network is the 1-complex, dual to the boundary of the 4-simplex, having five 4-valent vertices (corresponding to the five tetrahedra), with each of the ten edges connecting two different vertices (corresponding to the ten triangles of the 4-simplex each shared by two tetrahedra). Now we associate to each triangle (the dual of which is an edge) a simple representation of the algebra SO(4) and to each tetrahedra (the dual of which is a vertex) a intertwiner; and to a 4-simplex the product of the five intertwiner with the indices suitable contracted, and the sum for all possible representations. The proposed state sum suitable for quantum gravity for a given triangulation (decomposed into 4-simplices) is

$$Z_{BC} = \sum_{J \text{ triang.}} \prod_{\text{tetrahedra}} A_{\text{tetr.}} \prod_{\text{4-simplices}} A_{\text{simp.}}$$
 (38)

where the sum extends to all possible values of the representations J.

6. The triple product in \mathbb{R}^4

Before we apply the representation theory developed in previous sections to the Barrett-Crane model we introduce some geometrical properties based in the triple product that generalizes the vector (cross) product in \mathbb{R}^3 . Given three vectors in \mathbb{R}^4 , we define the triple product:

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$$= w \wedge u \wedge v,$$

$$u \wedge u \wedge v = u \wedge v \wedge u = v \wedge u \wedge u = 0.$$
(39)

If the vectors in \mathbb{R}^4 have cartesian coordinates

$$u = (u_1, u_2, u_3, u_4)$$
, $v = (v_1, v_2, v_3, v_4)$, $w = (w_1, w_2, w_3, w_4)$,

we define an orthonormal basis in \mathbb{R}^4

$$\hat{i} = (1, 0, 0, 0)$$
, $\hat{j} = (0, 1, 0, 0)$, $\hat{k} = (0, 0, 1, 0)$, $\hat{\ell} = (0, 0, 0, 1)$.

The triple product of these vectors satisfies

$$\hat{i} \wedge \hat{j} \wedge \hat{k} = -\hat{\ell}$$
, $\hat{j} \wedge \hat{k} \wedge \hat{\ell} = \hat{i}$, $\hat{k} \wedge \hat{\ell} \wedge \hat{i} = -\hat{j}$, $\hat{i} \wedge \hat{j} \wedge \hat{\ell} = \hat{k}$.

In coordinates the triple product is given by the determinant

$$u \wedge v \wedge w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} & \hat{\ell} \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}. \tag{40}$$

The scalar quadruple product is defined by

$$a \cdot (b \wedge c \wedge d) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = [abcd] = -[abdc] =$$

$$= -[acbd] = [acdb] \text{ and so on.}$$

$$(41)$$

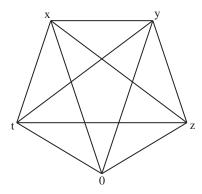
It follows: $a \cdot a \wedge b \wedge c = b \cdot a \wedge b \wedge c = c \cdot a \wedge b \wedge c = 0$.

We can use the properties of the three vector for the description of the 4-simplex. Let $\{0, x, y, z, t\}$ be the 4-simplex in \mathbb{R}^4 . Two tetrahedra have a common face

$$\{0, x, y, z\} \cap \{0, x, y, t\} = \{0, x, y\}.$$

Each tetrahedron is embedded in an hyperplane characterized by a vector perpendicular to all the vectors forming the tetrahedron. For instance,

 $\{0, x, y, z\}$ is characterized by $a = x \wedge y \wedge z$, $\{0, x, y, t\}$ is characterized by $b = x \wedge y \wedge t$.



The vector a satisfies $a \cdot x = a \cdot y = a \cdot z = 0$, the vector b satisfies $b \cdot x = b \cdot y = b \cdot t = 0$.

The triangle $\{0, x, y\}$ shared by the two tetrahedra is characterized by the bivector $x \wedge y$. The plane where the triangle is embedded is defined by the two vectors a, b, forming the angle ϕ , given by

$$\cos \phi = a \cdot b$$
.

The bivector $a \wedge b$ can be calculated with the help of trivectors as

$$a \wedge b = [x y z t]^* (x \wedge y).$$

Obviously $a \wedge b$ is perpendicular to $x \wedge y$

$$\langle a \wedge b, x \wedge y \rangle = (a \cdot x) (b \cdot y) - (a \cdot y) (b \cdot x) = 0.$$
 (42)

For completness we add some useful properties of bivectors in \mathbb{R}^4 . The six components of a bivector can be written as

$$\begin{array}{lll} B_{\mu\nu} = x_{\mu}y_{\nu} - x_{\nu}y_{\mu} & , & \mu, \nu = 1, 2, 3, 4 & , & B = (\bar{J}, \bar{K}) \, , \\ J_1 = (x_2y_3 - x_3y_2) & , & J_2 = (x_3y_1 - x_1y_3) & , & J_3 = (x_1y_2 - x_2y_1) \, , \\ K_1 = (x_1y_4 - x_4y_1) & , & K_2 = (x_2y_4 - x_3y_1) & , & K_3 = (x_3y_4 - x_4y_1) \, . \end{array}$$

The six components of the dual of a bivector are

*
$$B = (\bar{K}, \bar{J})$$
, * $B_{\alpha\beta} = \frac{1}{2}b_{\mu\nu} \, \varepsilon_{\mu\nu\alpha\beta}$.
We take the linear combinations of \bar{J}, \bar{K}

$$\bar{M} = \frac{1}{2} (\bar{J} + \bar{K}), \qquad \bar{N} = \frac{1}{2} (\bar{J} - \bar{K}).$$
 (43)

They form the bivector (\bar{M}, \bar{N}) , whose dual is:

$$^{*}(M,N) = (M,-N),$$
 (44)

therefore \bar{M} can be considered the self-dual part, \bar{N} the antiselfdual part of the bivector (\bar{M}, \bar{N}) . \bar{M} and \bar{N} coincides with the basis for the irreducible tensor representations of section 3. The norm of the bivectors can be explicitly calculated.

$$||B||^{2} = \langle B, B \rangle = J^{2} + K^{2} = ||x||^{2} ||y||^{2} - |x, y|^{2} =$$

$$= ||x||^{2} ||y||^{2} \sin^{2} \phi(x, y) = 4Area^{2} \{0, x, y\}, \qquad (45)$$

$$||^{*}B||^{2} = \langle ^{*}B, ^{*}B \rangle = J^{2} + K^{2} = ||B||^{2}. \qquad (46)$$

Finally, the scalar product of two vectors in \mathbb{R}^4 can be expressed in terms of the corresponding $U\left(2,\mathcal{C}\right)$ matrices

Let
$$X \Leftrightarrow \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$
, $Y \Leftrightarrow \begin{pmatrix} y_1 + iy_2 & y_3 + iy_4 \\ -y_3 + iy_4 & y_1 - iy_2 \end{pmatrix}$.

Then
$$Tr(X^+Y) = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4. \tag{47}$$

7. Evaluation of the spin sum for the relativistic spin network

The five tetrahedra in the 4-simplex are numbered by k = 1, 2, 3, 4, 5 and the triangles are indexed by the pair k, l of tetrahedra which intersect on the triangle kl. To each triangle we associate a simple representation of SO(4) labelled by j_{kl} , that corresponds to the same spin for each part of the $SU(2) \otimes SU(2)$ group. The matrix representing an element $g \in SU(2)$ in the irreducible representation of spin j_{kl} belonging to a triangle is denoted by $\rho_{kl}(g)$. An element $h_k \in SU(2)$ is assigned to each tetrahedron k. The invariant I is defined by integrating a function of these variables over each copy of SU(2):

$$I = (-1)^{\sum_{k < \ell} 2j_{k\ell}} \int_{h \in SU(2)^5} \prod \operatorname{Tr} \rho_{k\ell} \left(h_k h_{\ell}^{-1} \right)$$
 (48)

The geometrical interpretation of this formula given by Barrett [17] is that since the manifold SU(2) is isomorphic to S^3 , each variable $h \in SU(2)$ can be regarded as a unit vector in \mathbb{R}^4 . This unit vector can be identified with the vector perpendicular to the hyperplane where the tetrahedron is embedded. The two variables h_k, h_l correspond in this picture to the two vectors a, b that were defined in the last section.

In our opinion there is some disagreement between the conditions given in Ref. [16] and the application of formula (2.1) in Ref. [17]. In the former paper an irreducible representation of SO(4) with two labels $j_1 = j_2$ is assigned to each triangle in the 4-simplex. In the last paper, a representation of SU(2) is assigned to each triangle. Therefore we

have the standard values for the trace of a general representation of the group SU(2) with spin j, namely, $\sin(2j+1)\phi/\sin\phi$ (Formula 4.1 of Ref. 17).

The disagreement can be avoided if one takes the trace with respect to the irreducible representation of SO(4) as described in Sections 3 and 4, where the parameters of the group SO(4) are the 3+3 cartesian independent coordinates of the two unit vectors h_k , h_ℓ , as defined before, or the 6 rotation angles of formula (28). In the last case we choose a system of reference for R_4 such that one unit vector corresponding to h_k , say a, has components (0,0,0,1) and the other one h_ℓ , say b, is located in the plane (x_3x_4) forming an angle τ with the first vector. In this particular situation all the rotation angles $\alpha = \beta = \gamma = \vartheta = \varphi = 0$ and the representation is restricted to $S_3(\tau)$.

From (31) and (32) the general element representation of SO(4) is restricted to

$$D_{j'm'jm}^{j_1j_2}(0,0,\tau,0,0,0) = d_{j'jm}^{j_1j_2}(\tau) \equiv d_{j'jm}^{[j_+,j_-]}(\tau). \tag{49}$$

In the case of a simple representations of SO(4) $j_{-}=j_{1}-j_{2}=0$, and the trace becomes

$$\operatorname{tr} D_{j'm'jm}^{[j_{+},0]}(\tau) = \sum_{j=0}^{j_{+}} \sum_{m=-j}^{j} d_{jjm}^{[j_{+},0]}(\tau). \tag{50}$$

Obviously this expression does not coincide with formula (4.1) of Ref. [17] except in the term

$$d_{000}^{[j_{+},0]}(\tau) = \frac{1}{j_{+}+1} \frac{\sin((j_{+}+1)\tau)}{\sin \tau}, \text{ Ref [15], (IV.2.9)}.$$

For other values of the Biedenharn-Dolginov function we can use the asymptotic expression (37) for m = j. With this formula it is still possible to give an geometrical interpretation of the probability amplitude encompassed in the trace. In fact, the spin dependent factor appearing in the exponential of (37)

$$e^{i(2j_{k\ell}+1)\tau_{k\ell}},\tag{51}$$

corresponding to two tetrahedra $k\ell$ intersecting the triangle $k\ell$, can be interpreted as the product of the angle between the two vectors h_k, h_ℓ perpendicular to the triangle and the area $A_{k\ell}$ of the intersecting triangle.

For the proof we identify the component of the antisymmetrie tensor (\bar{J}, \bar{K}) with the components of the infinitesimal generators of the SO(4)

group

$$J_{\mu\nu} \equiv i \left(x_{\mu} \frac{\partial}{\partial x_{\nu}} - x_{\nu} \frac{\partial}{\partial x_{\mu}} \right).$$

From (43) and (45) we have $||B||^2 = 4 (A_{k\ell})^2 = 2 (\bar{M}^2 + \bar{N}^2)$ But \bar{M}^2 and \bar{N}^2 are the Casimir operators of the $SU(2) \otimes SU(2)$ group with eigenvalues $j_1(j_1 + 1)$ and $j_2(j_2 + 1)$.

For large values of $j_1 = j_2 = j_{k\ell}$ we have

$$2\left(\bar{M}^2 + \bar{N}^2\right) \cong 4j_{k\ell}^2 + 4j_{k\ell} + 1 = (2j_{k\ell} + 1)^2$$

therefore $\frac{1}{2}(2j_{k\ell}+1)=A_{k\ell}$ where $A_{k\ell}$ is the area of the triangle characterized by the two vectors h_k and h_ℓ and $j_{k\ell}$ is the spin corresponding to the representation $\rho_{k\ell}$ associated to the triangle kl. Substituting this result in (51) we obtain the assimptotic value of the amplitude given by Barrett and Williams [18].

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